The Lasserre Rank of the Cropped Hypercube

Gérard Cornuéjols, Vrishabh Patil, Jiaye Wei

September 23, 2026

Abstract

In an n-dimensional *cropped hypercube* each of the 2^n cropping inequalities chops off a single corner of the 0–1 hypercube by an ℓ_1-distance ρ. The case $\rho = 1/2$ has been extensively studied in the literature. This paper shows that the Lasserre rank of the n-dimensional cropped hypercube where $\rho = 1/2$, $n \geq 2$, is the smallest integer $0 \leq t \leq n$ such that $\Delta_t < 0$ in the recurrence $\Delta_{-1} = 1$, $\Delta_0 = n-1$, $\Delta_t = (n-1)\Delta_{t-1} - t(n-t+1)\Delta_{t-2}$. It follows that the Lasserre rank can be computed in time $O(n^2 \log^2 n)$. Asymptotically, the rank is $\frac{n}{2} + c_{1/2}\sqrt{n} + o(\sqrt{n})$, where $c_{1/2}$ is the unique zero of a given function. Numerically, $c_{1/2} \approx 0.3825$. In fact, we prove such results for any fixed $0 < \rho < 1$.

Keywords: Lasserre hierarchy; sum of squares; semidefinite programming; cropped hypercube; Krawtchouk polynomials

1 Introduction

Fix $0 < \rho < 1$ and let $N := \{1, \ldots, n\}$. The n-dimensional *cropped hypercube* is the polytope

$$Q_{n,\rho} := \Big\{ x \in [0,1]^n : g_I(x) \geq 0 \text{ for all } I \subseteq N \Big\},$$

where

$$g_I(x) := \sum_{i \in I} x_i + \sum_{i \in N \setminus I} (1 - x_i) - \rho. \tag{1}$$

Geometrically, each vertex of the 0-1 hypercube is cut off by a single inequality in the description of $Q_{n,\rho}$ at ℓ_1-distance ρ. Together, the 2^n inequalities exclude all 2^n vertices.

Writing $S_{n,\rho} := Q_{n,\rho} \cap \{0,1\}^n$, we have $S_{n,\rho} = \emptyset$ and therefore $\operatorname{conv}(S_{n,\rho}) = \emptyset$.

The cropped hypercube has been used as a canonical example for comparing cutting-plane and lift-and-project procedures. In particular, the cropped hypercube obtained by taking $\rho = 1/2$ has been studied extensively in the literature. The *rank* is the number of rounds of the procedure needed to reach $\operatorname{conv}(P \cap \mathbb{Z}^n)$ starting from a polyhedron P. The Chvátal–Gomory rank of $Q_{n,1/2}$ is n (Bockmayr et al. (1999)). Laurent (2003) showed that the Sherali–Adams hierarchy also requires n rounds for $Q_{n,1/2}$ whereas the Lasserre rank is at most $n - 1$. For general polytopes, however, Cheung (2007) established that the Lasserre rank is n in the worst case. Determining the precise Lasserre rank of the cropped hypercube has since become a natural test case for the strength of sum-of-squares relaxations. Kurpisz et al. (2016b) proved lower and upper bounds of $\Omega(\sqrt{n})$ and $n - \Omega(n^{1/3})$, respectively. Kurpisz (2019) sharpened the bounds for $Q_{n,1/2}$ as follows.

$$\left\lceil \frac{n}{2} \right\rceil \leq \operatorname{rank}_{\mathrm{LAS}}(Q_{n,1/2}) \leq \left\lceil \frac{n}{2} + \sqrt{n \log(2n)} \right\rceil.$$

More recently, Kurpisz and Wirth (2023) established a lower bound of $n/2 + \Omega(\sqrt{n})$. Their work reduces the question of identifying the Lasserre rank of $Q_{n,1/2}$ to one quadratic inequality and then to positive-definiteness conditions for a tridiagonal matrix. These conditions yield lower and upper bounds on the rank but do not identify the exact transition for every n. Symmetry reductions and univariate polynomial techniques have also been used more generally to analyze sum-of-squares certificates over the 0–1 hypercube (Kurpisz et al. (2016a, 2026)).

Au and Tunçel (2018) studied the Lasserre rank of cropped hypercubes when ρ decreases as n increases. They show that the rank is n when ρ goes to 0 sufficiently fast.

In this paper, our two main contributions are:

1. a simple recurrence for computing the Lasserre rank of $Q_{n,\rho}$. For every fixed rational ρ, the rank can be computed in $O(n^2 \log^2 n)$ time.

2. the following asymptotic formula for fixed $0 < \rho < 1$

$$\operatorname{rank}_{\mathrm{LAS}}(Q_{n,\rho}) = \frac{n}{2} + c_\rho\sqrt{n} + o(\sqrt{n}), \tag{2}$$

   where c_ρ is the unique zero of the function $\int_0^\infty s^\rho e^{-s^2/8} \cos\left(cs + \frac{\pi\rho}{2}\right)\, ds$. For $\rho = 1/2$, we have $c_{1/2} \approx 0.3825$. This closes the previous $\sqrt{\log n}$ gap in the second-order term and determines its sharp constant.

3. an exact characterization of the rank n regime. There is a unique threshold $\rho_n \in (0, 1)$ satisfying

$$\rho_n \sum_{r=1}^{n} \frac{\binom{n}{r}}{r - \rho_n} = 1,$$

   such that $\operatorname{rank} \mathrm{LAS}(Q_{n,\rho}) = n$ if and only if $0 < \rho \le \rho_n$. Moreover, $\rho_n \sim n/2^{n+1}$, closing the asymptotic factor-of-two gap between the previous bounds.

These results are obtained through a sequence of reductions from an exponentially indexed semidefinite system to a weighted least-squares problem. At level t, the Lasserre relaxation is empty if and only if this least-squares problem with t variables has optimum strictly below ρ. This gives a different route to the tridiagonal structure identified by Kurpisz and Wirth (2023). Our first new step is to prove that the sign of the least-squares optimum is exactly the sign of the determinant of an explicit tridiagonal matrix, which can be computed using a three-term recurrence. Relating this recurrence with Krawtchouk polynomials then allows us to determine the exact Lasserre rank for every n and $0 < \rho < 1$. The least-squares formulation can be solved directly for n up to 250. For example, when $\rho = 1/2$, the ranks for $n = 10$, 100, and 250 are 6, 53, and 131, respectively. The recurrence makes much larger computations routine. Still with $\rho = 1/2$, for $n = 10^3$, 10^4, and 10^5, the corresponding ranks are 512, 5038, and 50120, respectively. It is worth noting that Krawtchouk polynomials were already used by Slot and Laurent (2021) in the context of the Lasserre hierarchy.

The remainder of the paper is organized as follows. Section 2 recalls the Lasserre hierarchy. Section 3 uses the symmetries of the cropped hypercube to reduce feasibility to one canonical lifted point and one cropping inequality. Section 4 derives the least-squares characterization. Section 5 derives the tridiagonal and Krawtchouk representations and analyzes the critical sign transition. Section 6 presents the asymptotic analysis.

2 The Lasserre Hierarchy

We recall the Lasserre hierarchy for polyhedra described by linear inequalities (Lasserre (2001); Laurent (2003)). Let

$$P := \{x \in [0,1]^n : g_\ell(x) \geq 0,\ \ell = 1,\dots,m\}, \qquad S := P \cap \{0,1\}^n,$$

where each g_ℓ is a linear function. The lifted variables are $y = (y_I)_{I \subseteq N}$, intended to represent the monomials $\prod_{i \in I} x_i$. Since $x_i^2 = x_i$ for $x_i \in \{0,1\}$, the product of two monomials indexed by I and J is indexed by $I \cup J$.

For $r \geq 0$, the *moment matrix* of order r is indexed by the subsets of N of cardinality at most r and has entries

$$[M_r(y)]_{I,J} := y_{I \cup J}, \qquad |I|, |J| \leq r. \tag{3}$$

If $g(x) = b - \sum_{i=1}^n a_i x_i$ is linear, define the shifted sequence $g \cdot y$ by

$$(g \cdot y)_K := b y_K - \sum_{i=1}^n a_i y_{K \cup \{i\}}, \qquad K \subseteq N. \tag{4}$$

The matrix $M_r(g \cdot y)$ is the *localizing matrix* of g of order r. Its (I,J)-entry is

$$[M_r(g \cdot y)]_{I,J} = b y_{I \cup J} - \sum_{i=1}^n a_i y_{I \cup J \cup \{i\}}. \tag{5}$$

For an integer $t \geq 0$, the lifted level-t relaxation is

$$\widetilde{\mathrm{LAS}}_t(P) := \{y : y_\emptyset = 1, \quad M_{t+1}(y) \succeq 0, \quad M_t(g_\ell \cdot y) \succeq 0 \text{ for } \ell = 1,\dots,m\}, \tag{6}$$

and its projection onto the space of original variables is

$$\mathrm{LAS}_t(P) := \left\{x \in \mathbb{R}^n : x_i = y_{\{i\}} \text{ for all } i \in N \text{ and some } y \in \widetilde{\mathrm{LAS}}_t(P)\right\}. \tag{7}$$

We use the convention in which the moment matrix has order $t+1$ and the localizing matrices have order t. The bound inequalities $x_i \geq 0$ and $1 - x_i \geq 0$ need not be included among the g_ℓ. Their localizing-matrix conditions are implied by $M_{t+1}(y) \succeq 0$ (Laurent (2003), Lemma 5).

To see the meaning of these semidefinite conditions, fix $v \in \{0,1\}^n$ and set $\mu(v)_I := \prod_{i \in I} v_i$ for $I \subseteq N$. Then $M_r(\mu(v))$ is the rank-one matrix whose (I,J)-entry is $\mu(v)_I \mu(v)_J$, and

$$M_r(g \cdot \mu(v)) = g(v) M_r(\mu(v)).$$

Consequently, $\mu(v)$ is feasible for (6) whenever $v \in S$, and hence $\mathrm{conv}(S) \subseteq \mathrm{LAS}_t(P)$. The hierarchy is nested and satisfies $\mathrm{LAS}_n(P) = \mathrm{conv}(S)$ (Lasserre (2001); Laurent (2003)). Its *Lasserre rank* is the least t for which $\mathrm{LAS}_t(P) = \mathrm{conv}(S)$.

We assume that $n \geq 2$ for the remainder of the paper.

Figure 1 shows the three relaxations $\mathrm{LAS}_t(Q_{3,1/2})$ for $t = 0,1,2$. In particular, $\mathrm{LAS}_0(Q_{3,1/2}) = Q_{3,1/2}$, the nonempty spectrahedron $\mathrm{LAS}_1(Q_{3,1/2})$ contains $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, and $\mathrm{LAS}_2(Q_{3,1/2}) = \emptyset$. Of course, $\mathrm{LAS}_3(Q_{3,1/2}) = \emptyset$ as well.

Henceforth, we write $\widetilde{\mathrm{LAS}}_t$ and LAS_t for $\widetilde{\mathrm{LAS}}_t(Q_{n,\rho})$ and $\mathrm{LAS}_t(Q_{n,\rho})$, respectively. In the next section, we exploit the symmetries of $Q_{n,\rho}$ to decide this infeasibility question at a single canonical moment vector y^*.

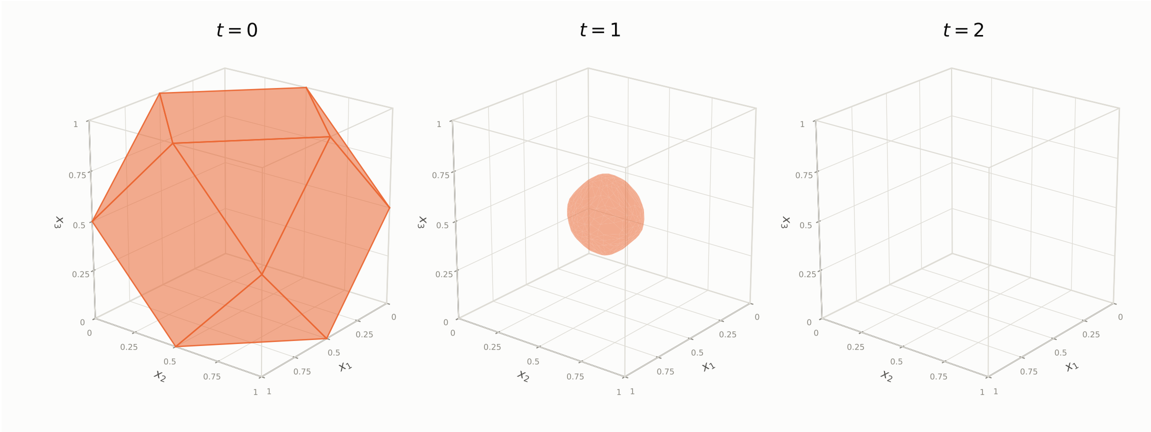

Figure 1: The Lasserre relaxations for $Q_{3,1/2}$ at levels $t = 0, 1, 2$.

3 Symmetry Reduction

We exploit the symmetries of the cropped hypercube to reduce the feasibility of the level-t Lasserre relaxation to a single lifted vector y^* and, subsequently, to a single cropping inequality. This isolates the only localizing matrix that needs to be analyzed.

Let Γ be the group generated by coordinate permutations and coordinate complementations $\tau_i : x_i \mapsto 1 - x_i$. A permutation σ maps g_I to $g_{\sigma(I)}$, while τ_i maps g_I to $g_{I \triangle \{i\}}$. Both operations also permute the bound inequalities. Thus, every element of Γ preserves the formulation.

We will use the following technical lemma to prove our main results. For an integer $k \geq 0$, let $m_k(x)$ denote the column vector of monomials $\prod_{i \in I} x_i$ with $I \subseteq N$ such that $|I| \leq k$.

Lemma 1. *Order the entries of $m_k(x)$ first by degree and then in any fixed order within each degree. For every $\gamma \in \Gamma$, there is an invertible matrix C_γ such that*

$$m_k(\gamma(x)) = C_\gamma m_k(x). \tag{8}$$

If γ is a coordinate permutation, then C_γ is a permutation matrix. If γ is a complementation of a set of coordinates, then C_γ is triangular with diagonal entries in $\{\pm 1\}$, and hence is invertible.

Proof. A coordinate permutation simply permutes the monomials of each degree, so it induces a permutation matrix on the entries of $m_k(x)$.

Now suppose that γ complements the coordinates in a set $I \subseteq N$. For a monomial indexed by $J \subseteq N$ with $|J| \leq k$, we have

$$\prod_{i \in J} \gamma(x)_i = \prod_{i \in J \cap I} (1 - x_i) \prod_{i \in J \setminus I} x_i = \sum_{T \subseteq J \cap I} (-1)^{|T|} \prod_{i \in (J \setminus I) \cup T} x_i.$$

Every monomial on the right has degree at most $|J|$. Therefore, with the degree ordering above, the matrix C_γ is triangular. The coefficient of the original monomial $\prod_{i \in J} x_i$ is $(-1)^{|J \cap I|}$, which is either 1 or -1. Thus all diagonal entries of C_γ are ± 1, so C_γ is invertible because the determinant of a triangular matrix is the product of its diagonal entries. $\square$

Define y^* by

$$y^*_I := \left(\frac{1}{2}\right)^{|I|}, \qquad I \subseteq N.$$

Note that $y^* = \sum_{v\in\{0,1\}^n} \lambda_v \mu(v)$ with equal weights $\lambda_v = 2^{-n}$ on all v, where $\mu(v)_I := \prod_{i\in I} v_i$ (recall the definition from the previous section).

For $\rho = 1/2$, Laurent showed that the vector y^* is feasible at level $n-1$ of the Sherali–Adams relaxation for the cropped hypercube (see Laurent (2003), Proposition 11). We prove the following.

Lemma 2. *The relaxation* LAS_t *is nonempty if and only if* y^* *belongs to* $\widetilde{\mathrm{LAS}}_t$*.*

Proof. One direction is straightforward. If $y^* \in \widetilde{\mathrm{LAS}}_t$, then its projection belongs to LAS_t, which is therefore nonempty.

For the converse, suppose that LAS_t is nonempty and let $y \in \widetilde{\mathrm{LAS}}_t$. Each $\gamma \in \Gamma$ induces an invertible linear map on the lifted variables by substituting $\gamma(x)$ into each monomial and replacing every resulting monomial indexed by I with y_I. Indeed, γ is a composition of a permutation and complementations: a permutation gives $(\sigma \cdot y)_I = y_{\sigma(I)}$, while a complementation gives

$$(\tau_i \cdot y)_I = \begin{cases} y_{I\setminus\{i\}} - y_I, & i \in I, \\ y_I, & i \notin I. \end{cases} \tag{9}$$

Both maps fix $y_\emptyset = 1$.

Claim 1. *The vector* $\bar{y} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma \cdot y$ *belongs to* $\widetilde{\mathrm{LAS}}_t$*.*

Proof of Claim. Observe that the feasible region $\widetilde{\mathrm{LAS}}_t$ is convex, and that it is invariant under $\gamma \in \Gamma$.

Indeed, γ permutes the 2^n inequalities g_I among themselves. For the matrix conditions themselves, $M_k(y)$ is the linearization of $m_k(x)\, m_k(x)^\top$. By Lemma 1, $m_k(\gamma(x)) = C_\gamma\, m_k(x)$ for an invertible matrix C_γ. Consequently,

$$M_k(\gamma \cdot y) = C_\gamma M_k(y) C_\gamma^\top.$$

Moreover, if $J \subseteq N$ is determined by $g_I(\gamma(x)) = g_J(x)$, then

$$M_k(g_I \cdot (\gamma \cdot y)) = C_\gamma M_k(g_J \cdot y) C_\gamma^\top.$$

For any positive semidefinite matrix M and any invertible matrix C, the matrix $CMC^\top$ is also positive semidefinite because $v^\top CMC^\top v = (C^\top v)^\top M (C^\top v) \geq 0$ for every v. Taking $k = t+1$ for the moment matrix and $k = t$ for the localizing matrices therefore shows that γ preserves all the semidefinite conditions in (6).

Hence $\gamma \cdot y$ is feasible for every $\gamma \in \Gamma$, and by convexity so is the average $\bar{y} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma \cdot y$. $\diamond$

It remains to identify $\bar{y}$, which satisfies $\bar{y}_\emptyset = 1$ and $\gamma \cdot \bar{y} = \bar{y}$ for every $\gamma \in \Gamma$, since averaging over all of Γ and then applying one more map reproduces the same average. Applying (9) with $i \in I$, that invariance forces $\bar{y}_I = \bar{y}_{I\setminus\{i\}} - \bar{y}_I$, that is $\bar{y}_I = \frac{1}{2}\bar{y}_{I\setminus\{i\}}$, and induction on $|I|$ from $\bar{y}_\emptyset = 1$ gives $\bar{y}_I = (\frac{1}{2})^{|I|}$. So $\bar{y} = y^*$, which is therefore feasible for $\widetilde{\mathrm{LAS}}_t$. $\square$

Consequently, the Lasserre rank of $Q_{n,\rho}$ is the smallest t at which y^* fails one of the level-t semidefinite conditions. This reduces the question to evaluating explicit matrices at a single rational vector.

We next show that the moment matrix alone cannot distinguish the cropped hypercube from its integer hull. For a vector α indexed by the subsets $\{I \subseteq N : |I| \leq k\}$, define the polynomial

$$p_\alpha(x) := \alpha^\top m_k(x) = \sum_{I \subseteq N : |I| \leq k} \alpha_I \prod_{i \in I} x_i.$$

Lemma 3. *$M_k(y^*) \succeq 0$ for every k and every n. More generally, if $g \geq 0$ at every point of $\{0,1\}^n$, then $M_k(g \cdot y^*) \succeq 0$ for every k.*

Proof. The definition of y^* gives

$$y^*_{I \cup J} = \sum_{v \in \{0,1\}^n} 2^{-n} \prod_{i \in I \cup J} v_i.$$

Fix any coefficient vector α indexed by $\{I : |I| \leq k\}$. Since $v_i^2 = v_i$ for $v \in \{0,1\}^n$, we have $\prod_{i \in I} v_i \prod_{j \in J} v_j = \prod_{i \in I \cup J} v_i$. It follows that

$$\alpha^\top M_k(y^*)\, \alpha = \sum_{I,J} \alpha_I \alpha_J \sum_v 2^{-n} \prod_{i \in I \cup J} v_i = \sum_v 2^{-n}\, p_\alpha(v)^2 \geq 0.$$

For the second claim, observe that for every $K \subseteq N$,

$$(g \cdot y^*)_K = \sum_{v \in \{0,1\}^n} 2^{-n} g(v) \prod_{i \in K} v_i.$$

The identical computation gives

$$\alpha^\top M_k(g \cdot y^*) \alpha = \sum_{v \in \{0,1\}^n} 2^{-n} g(v) p_\alpha(v)^2,$$

which is nonnegative whenever g is nonnegative at every point of $\{0,1\}^n$. $\square$

Thus, the moment-matrix condition cannot make y^* infeasible. The localizing matrices associated with inequalities valid for the 0–1 cube cannot make it infeasible either. In particular, the bound inequalities cannot cut off y^*. A cropping inequality $g_I(x) \geq 0$, however, is negative at the cube vertex that it cuts off. A certificate that the level-t relaxation is empty is therefore a vector α indexed by the subsets of cardinality at most t for which

$$\alpha^\top M_t(g_I \cdot y^*) \alpha = \sum_{v \in \{0,1\}^n} g_I(v) 2^{-n} p_\alpha(v)^2 < 0.$$

We can reduce the search for such a certificate α to a single cropping inequality as we will show in the next lemma. For $I \subseteq N$, let γ_I denote the coordinate map defined by

$$\gamma_I(x)_i := \begin{cases} x_i, & i \in I, \\ 1 - x_i, & i \in N \setminus I. \end{cases}$$

Applying γ_I twice returns every coordinate to its original value. Thus, γ_I is a self-inverse map on $\{0,1\}^n$, and

$$g_I(\gamma_I(x)) = \sum_{i\in I} x_i + \sum_{i\in N\setminus I} x_i - \rho = \sum_{i=1}^{n} x_i - \rho = g_N(x).$$

Since γ_I is a complementation, Lemma 1 gives an invertible matrix C_{γ_I} such that $m_k(\gamma_I(x)) = C_{\gamma_I}\, m_k(x)$. Moreover, the inverse of γ_I is itself, so applying (8) twice gives

$$m_k(x) = m_k(\gamma_I(\gamma_I(x))) = C_{\gamma_I}^2\, m_k(x).$$

The entries of $m_k(x)$ form a basis for the space of multilinear polynomials of degree at most k. Therefore, the identity $m_k(x) = C_{\gamma_I}^2 m_k(x)$ implies that $C_{\gamma_I}^2$ is the identity matrix. In particular, $C_{\gamma_I}^{-1} = C_{\gamma_I}$.

Lemma 4. *For every $I \subseteq N$ and every order k, there exists a vector α such that*

$$\alpha^\top M_k(g_I \cdot y^*)\alpha < 0$$

if and only if there exists a vector $\widetilde{\alpha}$ such that

$$\widetilde{\alpha}^\top M_k(g_N \cdot y^*)\widetilde{\alpha} < 0,$$

where $g_N(x) = \sum_{i=1}^n x_i - \rho$.

Proof. We first derive a relation between the corresponding quadratic forms. Fix $I \subseteq N$ and let q be any vector indexed by $\{J \subseteq N : |J| \le k\}$. Then,

$$\begin{aligned}
q^\top M_k(g_N \cdot y^*)q &= 2^{-n} \sum_{v\in\{0,1\}^n} g_N(v)\big(q^\top m_k(v)\big)^2 \\
&= 2^{-n} \sum_{u\in\{0,1\}^n} g_I(u)\big(q^\top m_k(\gamma_I(u))\big)^2 \\
&= 2^{-n} \sum_{u\in\{0,1\}^n} g_I(u)\big((C_{\gamma_I}^\top q)^\top m_k(u)\big)^2 \\
&= (C_{\gamma_I}^\top q)^\top M_k(g_I \cdot y^*)(C_{\gamma_I}^\top q).
\end{aligned}$$

The second equality holds because γ_I is a bijection of $\{0,1\}^n$ and $g_N(\gamma_I(u)) = g_I(u)$.

Now suppose that α is a certificate for g_I, such that $\alpha^\top M_k(g_I \cdot y^*)\alpha < 0$, and set $\widetilde{\alpha} := C_{\gamma_I}^\top \alpha$. Since $C_{\gamma_I}^2$ is the identity matrix, $(C_{\gamma_I}^\top)^2$ is also the identity matrix, and hence $C_{\gamma_I}^\top \widetilde{\alpha} = \alpha$. Applying the equality above with $q = \widetilde{\alpha}$ gives

$$\widetilde{\alpha}^\top M_k(g_N \cdot y^*)\widetilde{\alpha} = \alpha^\top M_k(g_I \cdot y^*)\alpha < 0.$$

This proves that every certificate for g_I produces a certificate for g_N. For the reverse direction, suppose that $\widetilde{\alpha}$ is a certificate for g_N and set $\alpha := C_{\gamma_I}^\top \widetilde{\alpha}$. Applying the equality above with $q = \widetilde{\alpha}$ gives

$$\alpha^\top M_k(g_I \cdot y^*)\alpha = \widetilde{\alpha}^\top M_k(g_N \cdot y^*)\widetilde{\alpha} < 0.$$

□

It therefore suffices to consider the single cropping inequality $g_N(x) = \sum_{i=1}^n x_i - \rho$.

4 Reduction to a Least-Squares Problem

Having reduced the analysis to the single localizing matrix $M_t(g_N \cdot y^*)$, we now simplify the search for a negative certificate. We first show that it suffices to consider coefficient vectors that are constant on sets of the same cardinality. This reduces the quadratic form to an explicit t-dimensional optimization problem, which can then be written as a weighted least-squares problem.

Lemma 5. *There exists a vector α indexed by $\{I \subseteq N : |I| \leq t\}$ such that $\alpha^\top M_t(g_N \cdot y^*)\alpha < 0$ if and only if there exists such a vector $\widetilde{\alpha}$ satisfying*

$$\widetilde{\alpha}_I = \widetilde{\alpha}_J \qquad \text{whenever } |I| = |J|.$$

Proof. The reverse implication is immediate, so it suffices to prove the forward implication.

Given a certificate α, define

$$\widetilde{\alpha}_I := \frac{1}{\binom{n}{|I|}} \sum_{\substack{J \subseteq N \\ |J|=|I|}} \alpha_J, \qquad |I| \leq t.$$

By construction, $\widetilde{\alpha}_I$ depends only on $|I|$. Fix $r \in \{0, \ldots, n\}$ and let

$$V_r := \left\{ v \in \{0,1\}^n : \sum_{i=1}^n v_i = r \right\}.$$

For a set $I \subseteq N$ with $|I| = \ell$, the monomial $\prod_{i \in I} v_i$ equals one for exactly $\binom{n-\ell}{r-\ell}$ vectors $v \in V_r$. Therefore,

$$\begin{aligned}
\sum_{v \in V_r} p_\alpha(v) &= \sum_{\ell=0}^{t} \binom{n-\ell}{r-\ell} \sum_{I \subseteq N : |I| = \ell} \alpha_I \\
&= \sum_{\ell=0}^{t} \binom{n-\ell}{r-\ell} \sum_{I \subseteq N : |I| = \ell} \widetilde{\alpha}_I \\
&= \sum_{v \in V_r} p_{\widetilde{\alpha}}(v),
\end{aligned}$$

where $\binom{n-\ell}{r-\ell} = 0$ when $\ell > r$. Moreover,

$$\sum_{v \in V_r} p_{\widetilde{\alpha}}(v)^2 = \frac{1}{|V_r|} \left(\sum_{v \in V_r} p_{\widetilde{\alpha}}(v) \right)^2 = \frac{1}{|V_r|} \left(\sum_{v \in V_r} p_\alpha(v) \right)^2 \leq \sum_{v \in V_r} p_\alpha(v)^2,$$

where the first equality holds because $p_{\widetilde{\alpha}}(v)$ has the same value for every $v \in V_r$, the second equality holds because p_α and $p_{\widetilde{\alpha}}$ have the same sum over V_r, and the inequality is Cauchy–Schwarz.

For $r = 0$, equality holds because $\widetilde{\alpha}_\emptyset = \alpha_\emptyset$. Since $g_N(v) = r - \rho$ for every $v \in V_r$, and $r - \rho > 0$ for every $r \geq 1$, it follows that

$$\begin{aligned}
\widetilde{\alpha}^\top M_t(g_N \cdot y^*)\widetilde{\alpha} &= 2^{-n} \sum_{r=0}^{n} (r - \rho) \sum_{v \in V_r} p_{\widetilde{\alpha}}(v)^2 \\
&\leq 2^{-n} \sum_{r=0}^{n} (r - \rho) \sum_{v \in V_r} p_\alpha(v)^2 \\
&= \alpha^\top M_t(g_N \cdot y^*)\alpha.
\end{aligned}$$

Thus, if $\alpha^\top M_t(g_N \cdot y^*)\alpha < 0$, then also $\widetilde{\alpha}^\top M_t(g_N \cdot y^*)\widetilde{\alpha} < 0$. $\square$

We may further assume that $\alpha_\emptyset = 1$. Indeed, if $\alpha_\emptyset = 0$, then $p_\alpha(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the all-zero vector. Since $g_N(v) = |\operatorname{supp}(v)| - \rho > 0$ for every $v \in \{0,1\}^n \setminus \{\mathbf{0}\}$, we have

$$\alpha^\top M_t(g_N \cdot y^*)\alpha = 2^{-n} \sum_{v \in \{0,1\}^n} g_N(v) p_\alpha(v)^2 = 2^{-n} \sum_{v \neq \mathbf{0}} g_N(v) p_\alpha(v)^2 \geq 0.$$

Thus every negative certificate has $\alpha_\emptyset \neq 0$. Replacing α by $\alpha/\alpha_\emptyset$ preserves the negativity of the quadratic form and normalizes the certificate so that $\alpha_\emptyset = 1$.

By Lemma 5, we may assume that α_I depends only on $|I|$. Define $z \in \mathbb{R}^t$ by $z_j := \alpha_I$ for every $I \subseteq N$ with $|I| = j$, where $j = 1, \ldots, t$. If $v \in \{0,1\}^n$ satisfies $\sum_{i=1}^n v_i = r$, then exactly $\binom{r}{j}$ monomials of degree j evaluate to one at v. Consequently, $p_\alpha(v) = 1 + \sum_{j=1}^t \binom{r}{j} z_j$. Note that this expression is now stated in terms of z and r. It will be denoted by $p_z(r)$ in the next section.

There are $\binom{n}{r}$ vectors $v \in \{0,1\}^n$ satisfying $\sum_i v_i = r$, and for each such vector, $g_N(v) = r - \rho$. Therefore,

$$2^n \alpha^\top M_t(g_N \cdot y^*)\alpha = -\rho + \sum_{r=1}^n \binom{n}{r}(r-\rho)\left(1 + \sum_{j=1}^t \binom{r}{j} z_j\right)^2 =: f_{n,t}(z). \tag{10}$$

For $r = 1, \ldots, n$, define $w_r := \binom{n}{r}(r - \rho)$. For each t, let $A_t \in \mathbb{R}^{n \times t}$ and $b \in \mathbb{R}^n$ be defined by

$$(A_t)_{r,j} := \sqrt{w_r}\binom{r}{j}, \qquad b_r := -\sqrt{w_r}, \tag{11}$$

for $r = 1, \ldots, n$ and $j = 1, \ldots, t$. Then

$$(A_t z - b)_r = \sqrt{w_r}\left(1 + \sum_{j=1}^t \binom{r}{j} z_j\right),$$

so that $f_{n,t}(z) = \|A_t z - b\|_2^2 - \rho$. We can now state the least-squares characterization.

Theorem 1. *The Lasserre rank of $Q_{n,\rho}$ is the smallest integer t such that*

$$\min_{z \in \mathbb{R}^t} \|A_t z - b\|_2^2 < \rho,$$

where A_t and b are defined in (11).

Proof. By Lemmas 2, 3, and 4, the level-t Lasserre relaxation is empty if and only if there exists α such that $\alpha^\top M_t(g_N \cdot y^*)\alpha < 0$. By Lemma 5, we may restrict attention to vectors whose entries depend only on the cardinality of the indexing set. Moreover, every negative certificate has $\alpha_\emptyset \neq 0$, so we may normalize to $\alpha_\emptyset = 1$. Thus such certificates are parametrized by $z \in \mathbb{R}^t$, and by (10),

$$2^n \alpha^\top M_t(g_N \cdot y^*)\alpha = \|A_t z - b\|_2^2 - \rho.$$

Hence the level-t relaxation is empty if and only if

$$\min_{z \in \mathbb{R}^t} \|A_t z - b\|_2^2 < \rho.$$

The result follows by taking the smallest such t. $\square$

For $\rho = 1/2$, solving the least-squares problems in Theorem 1 gives Lasserre ranks 6, 53, and 131 for $n = 10$, 100, and 250, respectively. For larger values of n, the weights $w_r = \binom{n}{r}(r - \rho)$ grow rapidly, making the direct least-squares formulation numerically intractable.

5 A Recurrence for Computing the Lasserre Rank

As shown in Theorem 1, computing the Lasserre rank of the cropped hypercube $Q_{n,\rho}$ amounts to finding the first integer t for which the minimum of $f_{n,t}(z)$ is negative. Using (10), we have

$$f_{n,t}(z) = \sum_{r=0}^{n} \binom{n}{r} (r-\rho) p_z(r)^2, \tag{12}$$

where $p_z(0) = 1$ and, for $0 \le t \le n$ and $z \in \mathbb{R}^t$, p_z is the univariate polynomial

$$p_z(x) := 1 + \sum_{j=1}^{t} \binom{x}{j} z_j. \tag{13}$$

The polynomial p_z is the univariate version of the symmetric multivariate polynomial p_α introduced in Section 3. For the associated coefficient vectors α and z, we have $p_\alpha(v) = p_z(r)$ whenever $\sum_{i=1}^{n} v_i = r$. In the remainder of this paper, we will need to extend the formula $\binom{x}{j} := x(x-1)\cdots(x-j+1)/j!$ to nonintegral scalars x. When $t = 0$, the sum in (13) is empty and $p_z(x) = 1$.

Denote the minimum of $f_{n,t}$ by $f^*_{n,t} := \min_{z\in\mathbb{R}^t} f_{n,t}(z)$. The next theorem shows that the first negative value of $f^*_{n,t}$ can be identified through a scalar recurrence.

Theorem 2. *For every $n \ge 2$ and $0 < \rho < 1$, define*

$$\Delta_{-1} := 1, \qquad \Delta_0 := n - 2\rho, \qquad \Delta_t := (n-2\rho)\Delta_{t-1} - t(n-t+1)\Delta_{t-2} \quad (1 \le t \le n). \tag{14}$$

Then

$$\operatorname{rank}_{\mathrm{LAS}}(Q_{n,\rho}) = \min\{0 \le t \le n \mid \Delta_t < 0\}. \tag{15}$$

In particular, the exact rank can be computed using at most n recurrence steps.

As an example, consider $n = 3$ and $\rho = 1/2$. The recurrence in (14) gives

$$\Delta_0 = 2, \qquad \Delta_1 = 1, \qquad \Delta_2 = -6.$$

It follows from Theorem 2 that the Lasserre rank of $Q_{3,1/2}$ is 2. This matches the rank observed in Figure 1.

The endpoint $t = n$ in (15) is essential. For example, when $n = 2$ and $\rho = 1/4$, the recurrence gives $\Delta_0 = 3/2$, $\Delta_1 = 1/4$, and $\Delta_2 = -21/8$, so the rank is 2.

To derive Theorem 2, we relate the minimization problem $\min_{z\in\mathbb{R}^t} f_{n,t}(z)$ to the sign change of a determinant. We first introduce the underlying square matrix H_t, as well as a related family of polynomials (Krawtchouk (1929)).

Fix integers $n \ge 2$ and $0 \le t \le n$. Let H_t be the $(t+1)\times(t+1)$ symmetric tridiagonal matrix indexed by $0,\dots,t$ whose nonzero entries are

$$(H_t)_{jj} = n - 2\rho \qquad (0 \le j \le t), \tag{16}$$

$$(H_t)_{j,j+1} = (H_t)_{j+1,j} = -\sqrt{(j+1)(n-j)} \qquad (0 \le j \le t-1). \tag{17}$$

Let $0 \le j \le n$ be an integer. The *binary Krawtchouk polynomials of degree j* are defined as

$$K_j^n(x) := \sum_{i=0}^{j} (-1)^i \binom{x}{i}\binom{n-x}{j-i}. \tag{18}$$

Throughout this paper, we refer to these polynomials simply as Krawtchouk polynomials. They can be obtained using a generating function:

$$K_j^n(x) = [u^j](1-u)^x(1+u)^{n-x}, \qquad 0 \le j \le n, \tag{19}$$

where $[u^j]F(u)$ denotes the coefficient of u^j in the power series of F (see MacWilliams and Sloane (1977) Chapter 5, Section 7, p. 151). We work with the normalized polynomials for fixed n

$$\phi_j(x) := \frac{K_j^n(x)}{\sqrt{\binom{n}{j}}}, \qquad 0 \le j \le n.$$

The following properties from Chapter 5, Section 7, pp. 150–152 of MacWilliams and Sloane (1977) will be used throughout this section.

i. The polynomial ϕ_j has degree j and satisfies $\phi_j(0) = \sqrt{\binom{n}{j}}$.

   Consequently, $\phi_0, \dots, \phi_t$ form a basis of the vector space $\mathbb{R}_t[x]$ of real polynomials of degree at most t.

ii. The polynomials $\phi_0, \dots, \phi_n$ are orthonormal with respect to the binomial distribution.

$$\mathbb{E}_{X\sim \mathrm{Bin}(n,1/2)}[\phi_i(X)\phi_j(X)] = 2^{-n}\sum_{r=0}^{n}\binom{n}{r}\phi_i(r)\phi_j(r) = \delta_{ij}, \qquad 0 \le i, j \le n, \tag{20}$$

   where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

iii. The polynomials satisfy the three-term recurrence

$$(n-2x)\phi_j(x) = \sqrt{(j+1)(n-j)}\,\phi_{j+1}(x) + \sqrt{j(n-j+1)}\,\phi_{j-1}(x), \tag{21}$$

   for $0 \le j \le n$, where $\phi_{-1} := 0$. Note that when $j = n$, we set the term $\sqrt{(j+1)(n-j)}\,\phi_{j+1}(x) = 0$.

Define $\nu \in \mathbb{R}^{t+1}$ by $\nu_j := \sqrt{\binom{n}{j}}$ for $0 \le j \le t$.

Lemma 6. *For any integers $n \ge 2$ and $1 \le t \le n$,*

$$f^*_{n,t} = 2^{n-1}\min\left\{\beta^\top H_t \beta \;\middle|\; \nu^\top\beta = 1,\ \beta \in \mathbb{R}^{t+1}\right\}. \tag{22}$$

Proof.

Claim 1. *There is a bijection between $\mathbb{R}^t$ and the affine space $\{\beta \in \mathbb{R}^{t+1} \mid \nu^\top\beta = 1\}$.*

Proof of Claim. Given $z \in \mathbb{R}^t$, the basis property of $\phi_0, \dots, \phi_t$ gives a unique vector $\beta = (\beta_0, \dots, \beta_t)^\top \in \mathbb{R}^{t+1}$ such that

$$p_z(x) = \sum_{j=0}^{t}\beta_j\phi_j(x). \tag{23}$$

Since $p_z(0) = 1$ and $\phi_j(0) = \nu_j$, we have $\nu^\top\beta = 1$.

Conversely, let $\beta \in \mathbb{R}^{t+1}$ satisfy $\nu^\top\beta = 1$ and define $q_\beta(x) := \sum_{j=0}^{t}\beta_j\phi_j(x)$. Then $q_\beta \in \mathbb{R}_t[x]$ and $q_\beta(0) = 1$. The polynomials $1, \binom{x}{1}, \dots, \binom{x}{t}$ also form a basis of $\mathbb{R}_t[x]$, so there is a unique $z \in \mathbb{R}^t$ such that

$$q_\beta(x) = 1 + \sum_{j=1}^{t}\binom{x}{j}z_j.$$

Thus, $z \mapsto \beta$ gives the claimed bijection. $\Diamond$

For every $\beta \in \mathbb{R}^{t+1}$, define $q_\beta(x) := \sum_{j=0}^{t} \beta_j \phi_j(x)$.

Claim 2. *For any $0 \le t \le n$ and any $\beta \in \mathbb{R}^{t+1}$,*

$$\beta^\top H_t \beta = 2^{-n} \sum_{r=0}^{n} \binom{n}{r} (2r - 2\rho) q_\beta(r)^2. \tag{24}$$

Proof of Claim. Let $X \sim \mathrm{Bin}(n, 1/2)$. By the definition of the binomial distribution,

$$2^{-n} \sum_{r=0}^{n} \binom{n}{r} (2r - 2\rho) q_\beta(r)^2 = \mathbb{E}\big[(2X - 2\rho) q_\beta(X)^2\big].$$

Decomposing the factor $2X - 2\rho$ gives

$$\mathbb{E}\big[(2X - 2\rho) q_\beta(X)^2\big] = (n - 2\rho)\mathbb{E}\big[q_\beta(X)^2\big] - \mathbb{E}\big[(n - 2X) q_\beta(X)^2\big].$$

Equation (20) and the definition of q_β yield

$$\mathbb{E}\big[q_\beta(X)^2\big] = \sum_{i,j=0}^{t} \beta_i \beta_j \mathbb{E}[\phi_i(X)\phi_j(X)] = \sum_{j=0}^{t} \beta_j^2.$$

Multiplying (21) by $\phi_i(X)$ and taking expectations gives

$$\mathbb{E}\big[\phi_i(X)(n - 2X)\phi_j(X)\big] = \sqrt{(j+1)(n-j)}\,\delta_{i,j+1} + \sqrt{j(n-j+1)}\,\delta_{i,j-1}.$$

When $t = n$, for $j = n$ we use the terminal identity $(n - 2X)\phi_n(X) = \sqrt{n}\,\phi_{n-1}(X)$. It follows that

$$\begin{aligned}
\mathbb{E}\big[(n - 2X) q_\beta(X)^2\big] &= \sum_{i,j=0}^{t} \beta_i \beta_j \mathbb{E}\big[\phi_i(X)(n - 2X)\phi_j(X)\big] \\
&= 2 \sum_{j=0}^{t-1} \sqrt{(j+1)(n-j)}\,\beta_j \beta_{j+1},
\end{aligned}$$

where the second equality follows by reindexing one of the two identical sums. Therefore,

$$\begin{aligned}
\mathbb{E}\big[(2X - 2\rho) q_\beta(X)^2\big] &= (n - 2\rho) \sum_{j=0}^{t} \beta_j^2 - 2 \sum_{j=0}^{t-1} \sqrt{(j+1)(n-j)}\,\beta_j \beta_{j+1} \\
&= \beta^\top H_t \beta.
\end{aligned}$$

Combining this with the first equality proves (24). $\Diamond$

Finally, fix $z \in \mathbb{R}^t$ and let $\beta \in \mathbb{R}^{t+1}$ be the unique coefficient vector corresponding to z under the bijection in Claim 1. Since $p_z = q_\beta$, equations (12) and (24) give

$$f_{n,t}(z) = \sum_{r=0}^{n} \binom{n}{r} (r - \rho) p_z(r)^2 = 2^{n-1} \beta^\top H_t \beta.$$

Together with the bijection from Claim 1, this proves (22). $\square$

For fixed n, the sequence $(f^*_{n,t})_{t=0}^n$ is nonincreasing in t. Indeed, for $t \le n-1$, appending a zero to any $z \in \mathbb{R}^t$ leaves both p_z and the objective value unchanged at level $t+1$.

The next lemma relates the sign of $f^*_{n,t}$ to that of $\det(H_t)$. We use

$$\operatorname{sgn}(a) := \begin{cases} -1, & a < 0, \\ 0, & a = 0, \\ 1, & a > 0. \end{cases}$$

Lemma 7. *For $n \ge 2$ and $1 \le t \le n$, we have*

$$\operatorname{sgn}(f^*_{n,t}) = \operatorname{sgn}(\det(H_t)).$$

Proof. We first show that H_t is positive definite on $\nu^\perp := \{d \in \mathbb{R}^{t+1} \mid \nu^\top d = 0\}$. Fix $0 \ne d \in \nu^\perp$. Recall the definition of $q_d(x)$:

$$q_d(x) := \sum_{j=0}^{t} d_j \phi_j(x).$$

Since $\phi_j(0) = \sqrt{\binom{n}{j}} = \nu_j$, we have $q_d(0) = \nu^\top d = 0$. Therefore,

$$d^\top H_t d = 2^{-n} \sum_{r=0}^{n} \binom{n}{r} (2r - 2\rho) q_d(r)^2 = 2^{-n} \sum_{r=1}^{n} \binom{n}{r} (2r - 2\rho) q_d(r)^2, \tag{25}$$

where the first equality is (24) and the second equality follows from $q_d(0) = 0$. Every coefficient in this sum is positive. If the sum were zero, then q_d would vanish at $0, 1, \ldots, n$. Since $\deg(q_d) \le t \le n$, this would imply $q_d \equiv 0$. The linear independence of $\phi_0, \ldots, \phi_t$ would then imply $d = 0$, which is a contradiction. Hence $d^\top H_t d > 0$ for every $0 \ne d \in \nu^\perp$.

Let $U \in \mathbb{R}^{(t+1)\times t}$ have columns that form a basis of $\nu^\perp$. Then $G := U^\top H_t U$ is positive definite by (25). Take, for example, the feasible point $\bar\beta := \nu/(\nu^\top \nu)$ in $\nu^\top \beta = 1$. Every feasible point in $\nu^\top \beta = 1$ can be written uniquely as $\bar\beta + U\xi$ for some $\xi \in \mathbb{R}^t$. Under this parametrization, the objective in (22) becomes

$$(\bar\beta + U\xi)^\top H_t (\bar\beta + U\xi) = \xi^\top G \xi + 2h^\top \xi + \bar\beta^\top H_t \bar\beta,$$

where $h := U^\top H_t \bar\beta$. Since G is positive definite, this quadratic function has the unique minimizer $\xi^* = -G^{-1}h$. Therefore, the problem in (22) has the unique minimizer $\beta^* := \bar\beta + U\xi^*$.

The first-order optimality conditions for the Lagrangian $\mathcal{L}(\beta, \lambda) = \beta^\top H_t \beta - 2\lambda(\nu^\top \beta - 1)$ give

$$H_t \beta^* = \lambda \nu$$

for some $\lambda \in \mathbb{R}$. Since $\nu^\top \beta^* = 1$, we obtain $(\beta^*)^\top H_t \beta^* = \lambda$. Equation (22) shows that $\operatorname{sgn}(f^*_{n,t}) = \operatorname{sgn}(\lambda)$

Define $P := [\, \beta^* \; U \,] \in \mathbb{R}^{(t+1)\times(t+1)}$. The matrix P is invertible because $\beta^* \notin \nu^\perp$ and the columns of U form a basis of $\nu^\perp$. Moreover,

$$P^\top H_t P = \begin{pmatrix} (\beta^*)^\top H_t \beta^* & (\beta^*)^\top H_t U \\ U^\top H_t \beta^* & U^\top H_t U \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & G \end{pmatrix},$$

where $U^\top H_t \beta^* = \lambda U^\top \nu = 0$. Taking determinants yields

$$(\det P)^2 \det(H_t) = \lambda \det(G).$$

Both $(\det P)^2$ and $\det(G)$ are positive, which proves the result. $\square$

We next derive a recurrence for $\det(H_t)$ and relate it to a Krawtchouk polynomial.

Lemma 8. *Let* $\Delta_t := \det(H_t)$ *for* $0 \le t \le n$ *and set* $\Delta_{-1} := 1$. *Then* $\Delta_0 = n - 2\rho$ *and, for* $1 \le t \le n$,

$$\Delta_t = (n - 2\rho)\Delta_{t-1} - t(n - t + 1)\Delta_{t-2}. \tag{26}$$

Moreover, for $0 \le t \le n - 1$,

$$\det(H_t) = (t + 1)!K^n_{t+1}(\rho). \tag{27}$$

Proof. Expanding the determinant of the tridiagonal matrix H_t along its last row gives the standard determinant recurrence (Usmani (1994), pp. 413–414)

$$\Delta_t = (H_t)_{tt}\Delta_{t-1} - (H_t)^2_{t,t-1}\Delta_{t-2}.$$

Equations (16) and (17) give $(H_t)_{tt} = n - 2\rho$ and $(H_t)^2_{t,t-1} = t(n - t + 1)$, which proves (26).

The unnormalized Krawtchouk recurrence given in Chapter 5, Section 7, p. 151 of MacWilliams and Sloane (1977) is

$$(n - 2x)K^n_j(x) = (j + 1)K^n_{j+1}(x) + (n - j + 1)K^n_{j-1}(x).$$

For $0 \le t \le n - 1$, setting $x = \rho$ and $j = t$ yields

$$(t + 1)K^n_{t+1}(\rho) = (n - 2\rho)K^n_t(\rho) - (n - t + 1)K^n_{t-1}(\rho).$$

For $1 \le t \le n - 1$, multiplication by $t!$ gives

$$(t + 1)!K^n_{t+1}(\rho) = (n - 2\rho)t!K^n_t(\rho) - t(n - t + 1)(t - 1)!K^n_{t-1}(\rho).$$

The corresponding initial values are $K^n_0(\rho) = 1 = \Delta_{-1}$ and $K^n_1(\rho) = n - 2\rho = \Delta_0$. Thus, the quantities $(t + 1)!K^n_{t+1}(\rho)$ satisfy the same recurrence and initial conditions as Δ_t, so $\Delta_t = (t + 1)!K^n_{t+1}(\rho)$ for $0 \le t \le n - 1$. This proves (27). □

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Theorem 1 and Lemmas 7 and 8, for $1 \le t \le n$,

$$\operatorname{sgn}(f^*_{n,t}) = \operatorname{sgn}(\det(H_t)) = \operatorname{sgn}(\Delta_t).$$

The same conclusion holds for $t = 0$ because $f^*_{n,0} = 2^{n-1}(n - 2\rho) > 0$ and $\Delta_0 = n - 2\rho > 0$.

Finally, at $t = n$, choosing $z_j = (-1)^j$ gives

$$p_z(r) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} = 0, \qquad r = 1, \ldots, n.$$

Hence $f^*_{n,n} = -\rho < 0$, so the set on the right-hand side of (15) is nonempty. Theorem 1 now shows that the rank is precisely the first t for which $\Delta_t < 0$. This proves (15). □

To bound the bit complexity for fixed rational $\rho = p/q$, with positive integers $p < q$, set $\widehat{\Delta}_t := q^{t+1}\Delta_t$. The initial values are $\widehat{\Delta}_{-1} = 1$ and $\widehat{\Delta}_0 = qn - 2p$, and

$$\widehat{\Delta}_t = (qn - 2p)\widehat{\Delta}_{t-1} - q^2 t(n - t + 1)\widehat{\Delta}_{t-2}.$$

This scaling preserves signs and gives an integer recurrence. Let $B_t := \max_{-1\le j\le t} |\widehat{\Delta}_j|$. Since p and q are fixed and $t(n-t+1) \le n^2$, the recurrence gives $B_t \le C_{p,q} n^2 B_{t-1}$ for a constant $C_{p,q}$, and hence each $\widehat{\Delta}_t$ has $O(n \log n)$ bits. Using standard integer arithmetic, each recurrence step takes $O(n \log^2 n)$ time, so all $\widehat{\Delta}_t$ can be computed in $O(n^2 \log^2 n)$ time.

If we allow ρ to vary with n, the Lasserre rank can be equal to n. Let ρ_n denote the largest value of ρ for which $Q_{n,\rho}$ has Lasserre rank n. This small-ρ regime was previously studied in Au and Tunçel (2018); Kurpisz (2019). Using $1/\rho$ as the parameter, (Kurpisz, 2019, Theorem 12 and Lemma 13) obtained lower and upper bounds on the rank when $1/\rho \ge 2$. In particular, (Kurpisz, 2019, Corollary 4.2) showed that $Q_{n,2^{-(n+1)}}$ has Lasserre rank n. More directly, (Au and Tunçel (2018), Theorem 14) proved that

$$\frac{n+1}{2^{n+2}-n-3} \le \rho_n \le \frac{n}{2^{n+1}-2}.$$

Thus, $\rho_n = \Theta(n/2^n)$, but these bounds differ asymptotically by a factor of two. The following theorem characterizes ρ_n exactly and shows that the upper bound in Au and Tunçel (2018) is asymptotically tight.

Theorem 3. *For every $n \ge 2$, there is a unique $\rho_n \in (0,1)$ satisfying*

$$\rho_n \sum_{r=1}^{n} \frac{\binom{n}{r}}{r-\rho_n} = 1. \tag{28}$$

Moreover, $\mathrm{rank}_{\mathrm{LAS}}(Q_{n,\rho}) = n$ *if and only if* $0 < \rho \le \rho_n$*, and* $\rho_n \sim n/2^{n+1}$.

Proof. For $0 \le \rho < 1$, let $T_n(\rho) := \sum_{r=1}^n \binom{n}{r}/(r-\rho)$. By Theorem 1 and the monotonicity of $f^*_{n,t}$, the rank is n if and only if $f^*_{n,n-1} \ge 0$. Consider the matrix $A_{n-1} \in \mathbb{R}^{n\times(n-1)}$ and vector $b \in \mathbb{R}^n$ defined in (11), so that $f_{n,n-1}(z) = \|A_{n-1}z - b\|_2^2 - \rho$. The first $n-1$ rows of A_{n-1} form an invertible lower triangular matrix, and hence A_{n-1} has rank $n-1$. Define $\eta \in \mathbb{R}^n$ by $\eta_r := (-1)^r \sqrt{\binom{n}{r}/(r-\rho)}$. For every $j = 1, \ldots, n-1$, the binomial theorem gives

$$(A_{n-1}^\top \eta)_j = \sum_{r=j}^{n} (-1)^r \binom{n}{r}\binom{r}{j} = (-1)^j \binom{n}{j} \sum_{s=0}^{n-j} (-1)^s \binom{n-j}{s} = 0.$$

Thus $\ker(A_{n-1}^\top) = \mathrm{span}\{\eta\}$. Moreover, $\|\eta\|_2^2 = T_n(\rho)$ and $\eta^\top b = -\sum_{r=1}^n (-1)^r \binom{n}{r} = 1$. If z^* minimizes $f_{n,n-1}$, the normal equations imply that $A_{n-1}z^* - b = \lambda\eta$ for some $\lambda \in \mathbb{R}$. Taking the inner product with η yields $\lambda T_n(\rho) = -1$, and therefore

$$f^*_{n,n-1} = \lambda^2 \|\eta\|_2^2 - \rho = \frac{1}{T_n(\rho)} - \rho.$$

Thus the Lasserre rank is n if and only if $\rho T_n(\rho) \le 1$. Since $\rho T_n(\rho)$ is continuous and strictly increasing from zero to infinity on $(0,1)$, this proves the existence and uniqueness of ρ_n and the claimed characterization of $\mathrm{rank}_{\mathrm{LAS}}(Q_{n,\rho}) = n$.

It remains to estimate ρ_n. Since $r(1-\rho_n) \le r - \rho_n \le r$ for every $r \ge 1$, we have $T_n(0) \le T_n(\rho_n) \le T_n(0)/(1-\rho_n)$. Together with $\rho_n T_n(\rho_n) = 1$, this gives

$$1 - \rho_n \le \rho_n T_n(0) \le 1.$$

Also, $T_n(0) \geq n$ implies $\rho_n \leq T_n(0)^{-1} \leq 1/n$, so $\rho_n \to 0$. Hence $\rho_n T_n(0) \to 1$, or equivalently, $\rho_n \sim T_n(0)^{-1}$. To estimate $T_n(0)^{-1}$, let $X_n \sim \mathrm{Bin}(n, 1/2)$. Then

$$T_n(0) = 2^n \mathbb{E}\big[X_n^{-1} \mathbf{1}_{\{X_n \geq 1\}}\big]\,,$$

where the expression inside the expectation is defined to be zero when $X_n = 0$. Set $E_n := \{|X_n - n/2| \leq n^{3/4}\}$ and let E_n^c denote the complement set. A Chernoff bound gives $\Pr(E_n^c) \leq 2e^{-2\sqrt{n}}$. On E_n, we have $X_n^{-1} = (2/n)(1+o(1))$ uniformly, and hence $\mathbb{E}[X_n^{-1}\mathbf{1}_{E_n}] = (2/n)(1+o(1))$. On the other hand, $X_n^{-1} \leq 1$ whenever $X_n \geq 1$, so

$$0 \leq \mathbb{E}\big[X_n^{-1}\mathbf{1}_{\{X_n \geq 1\} \cap E_n^c}\big] \leq \Pr(E_n^c) \leq 2e^{-2\sqrt{n}} = o(n^{-1}).$$

Consequently, $T_n(0) \sim 2^{n+1}/n$, and thus $\rho_n \sim T_n(0)^{-1} \sim n/2^{n+1}$. □

6 Asymptotically Tight Lasserre Rank

We next turn to the asymptotic analysis for fixed ρ. The cropping constant is fixed independently of n in the following theorem.

Theorem 4. *For every fixed $0 < \rho < 1$,*

$$\mathrm{rank}_{\mathrm{LAS}}(Q_{n,\rho}) = \frac{n}{2} + c_\rho \sqrt{n} + o(\sqrt{n}), \tag{29}$$

where $c_\rho > 0$ is the unique zero of the function

$$\kappa_\rho(c) := \int_0^\infty s^\rho e^{-s^2/8} \cos\Big(cs + \frac{\pi\rho}{2}\Big)\, ds. \tag{30}$$

In particular, $c_{1/2} \approx 0.3825$.

Lemmas 7 and 8 reduce the rank computation to the sign of $K_{t+1}^n(\rho)$ for $0 \leq t \leq n-1$. The asymptotic behavior of Krawtchouk polynomials has been studied in several parameter ranges (Dai and Wong (2007); Dominici (2008); Ismail and Simeonov (1998); Li and Wong (2000); Qiu and Wong (2004)). The proof of the following lemma is fairly standard but we could not find a good reference for it, so we include it for completeness.

Lemma 9. *For $n \geq 2$ and $0 \leq m \leq n$,*

$$K_m^n(\rho) = \frac{2^{n+1}}{\pi} \int_0^{\pi/2} (\cos\theta)^{n-\rho} (\sin\theta)^\rho \, \cos\Big((n-2m)\theta - \frac{\pi\rho}{2}\Big)\, d\theta. \tag{31}$$

Proof. The integral representation of the Krawtchouk polynomials (Johansson, 2002, p. 232, Eq. (2.7) with $p = 1/2$ and $z = -2u$) is

$$K_m^n(x) = \frac{1}{2\pi i} \oint_{|u|=R} (1+u)^{n-x}(1-u)^x \frac{du}{u^{m+1}}, \qquad 0 < R < 1.$$

Set $x = \rho$ and parametrize by $u = Re^{i\varphi}$. The factors in the integrand extend continuously to the boundary of the unit disk, and the parametrized integrand is uniformly bounded for $R \geq 1/2$. Thus, by dominated convergence, letting $R \uparrow 1$ gives

$$K_m^n(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+e^{i\varphi})^{n-\rho}(1-e^{i\varphi})^\rho \, e^{-im\varphi}\, d\varphi.$$

For $0 < \varphi < \pi$, we have $1 + e^{i\varphi} = 2\cos(\varphi/2)e^{i\varphi/2}$ and $1 - e^{i\varphi} = 2\sin(\varphi/2)e^{i(\varphi/2-\pi/2)}$. Pairing the conjugate integrands at φ and $-\varphi$ yields twice the real part, hence

$$K_m^n(\rho) = \frac{2^n}{\pi} \int_0^\pi \cos^{n-\rho}(\varphi/2) \sin^\rho(\varphi/2) \cos\left(\frac{n-2m}{2}\varphi - \frac{\pi\rho}{2}\right) d\varphi.$$

Substituting $\varphi = 2\theta$ proves (31). □

Next we focus on the case when $m = \frac{n}{2} + c_n\sqrt{n}$ where $c_n \in \mathbb{R}$.

Lemma 10. *Let $m = m(n) \in \{0, \dots, n\}$ for every integer $n \geq 2$, and write*

$$m = \frac{n}{2} + c_n\sqrt{n}, \qquad c_n \in \mathbb{R}.$$

If $c_n \to c \in \mathbb{R}$, then

$$\pi 2^{-n+\rho} n^{(\rho+1)/2} K_m^n(\rho) \longrightarrow \kappa_\rho(c), \tag{32}$$

where κ_ρ is the function (30)*. Moreover, κ_ρ has a unique zero c_ρ, and*

$$\operatorname{sgn}(\kappa_\rho(c)) = \operatorname{sgn}(c_\rho - c).$$

Proof. Since $n - 2m = -2c_n\sqrt{n}$, substituting $\theta = s/(2\sqrt{n})$ in (31) gives

$$\pi 2^{-n+\rho} n^{(\rho+1)/2} K_m^n(\rho) = \int_0^{\pi\sqrt{n}} s^\rho \left(\cos\frac{s}{2\sqrt{n}}\right)^{n-\rho} \left(\frac{\sin(s/(2\sqrt{n}))}{s/(2\sqrt{n})}\right)^\rho \cos\left(c_n s + \frac{\pi\rho}{2}\right), ds.$$

We interpret the sine quotient as one at $s = 0$ and extend the integrand by zero outside $[0, \pi\sqrt{n}]$. For every fixed $s \geq 0$, the integrand converges to

$$s^\rho e^{-s^2/8} \cos\left(cs + \frac{\pi\rho}{2}\right).$$

On $0 \leq x < \frac{\pi}{2}$, the inequalities $0 \leq \frac{\sin x}{x} \leq 1$ and $\log(\cos x) \leq -x^2/2$ imply

$$\left(\cos\frac{s}{2\sqrt{n}}\right)^{n-\rho} \leq \exp\left(-\frac{n-\rho}{8n}s^2\right) \leq e^{-s^2/16}$$

for $n \geq 2$. Thus, the absolute value of the extended integrand is bounded by the integrable function $s^\rho e^{-s^2/16}$. Therefore, (32) follows from the dominated convergence theorem.

The cosine integral representation of the parabolic-cylinder function D_ν in (Erdélyi et al., 1953, Section 8.3, Eq. (4), p. 120), applied with $\nu = \rho$, $z = -2c$, and the change of variables $s = 2r$, gives

$$\kappa_\rho(c) = 2^\rho \sqrt{2\pi}\, e^{-c^2} D_\rho(-2c).$$

For $0 < \rho < 1$, the function D_ρ has exactly one real zero (Erdélyi et al., 1953, Section 8.6, p. 126); denote it by x_ρ^*. The same integral representation at $z = 0$ shows that $D_\rho(0) > 0$. By (Erdélyi et al., 1953, Section 8.4, Eq. (2), p. 123), $D_\rho(-y) < 0$ for all sufficiently large y. Therefore by continuity we get $x_\rho^* < 0$ and

$$D_\rho(x) < 0 \quad \text{for } x < x_\rho^*, \qquad D_\rho(x) > 0 \quad \text{for } x > x_\rho^*.$$

Set $c_\rho := -x_\rho^*/2 > 0$. The factor $2^\rho\sqrt{2\pi}\, e^{-c^2}$ is positive, which gives the desired sign relation $\operatorname{sgn}(\kappa_\rho(c)) = \operatorname{sgn}(c_\rho - c)$. □

We now prove Theorem 4.

Proof of Theorem 4. It remains to prove (29). Write $r_n := \mathrm{rank}_{\mathrm{LAS}}(Q_{n,\rho})$. Fix $\varepsilon > 0$ and, for all sufficiently large n, set

$$m_n^- := \left\lfloor \frac{n}{2} + (c_\rho - \varepsilon)\sqrt{n} \right\rfloor, \qquad m_n^+ := \left\lceil \frac{n}{2} + (c_\rho + \varepsilon)\sqrt{n} \right\rceil.$$

These integers lie in $\{1, \dots, n\}$ for all sufficiently large n. Since rounding changes each value by at most 1, we have

$$\frac{m_n^- - n/2}{\sqrt{n}} \longrightarrow c_\rho - \varepsilon, \qquad \frac{m_n^+ - n/2}{\sqrt{n}} \longrightarrow c_\rho + \varepsilon.$$

Lemma 10 therefore gives

$$K^n_{m_n^-}(\rho) > 0 \qquad \text{and} \qquad K^n_{m_n^+}(\rho) < 0$$

for all sufficiently large n. By Lemmas 7 and 8, these inequalities imply $f^*_{n,m_n^- -1} > 0$ and $f^*_{n,m_n^+ -1} < 0$. Since $f^*_{n,t}$ is nonincreasing in t and r_n is the first level at which this minimum is negative, we obtain

$$m_n^- \le r_n \le m_n^+ - 1.$$

Consequently,

$$c_\rho - \varepsilon \le \liminf_{n\to\infty} \frac{r_n - n/2}{\sqrt{n}} \le \limsup_{n\to\infty} \frac{r_n - n/2}{\sqrt{n}} \le c_\rho + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ proves (29). □

Some numerical values are

ρ	99/100	9/10	3/4	3/5	1/2	1/4	1/10	1/100
c_ρ	0.006286	0.064705	0.170732	0.291092	0.382475	0.682246	0.991827	1.528217

The dependence of c_ρ on ρ is illustrated in Figure 2. The curve is nonlinear over $(0, 1)$ and exhibits different behavior at its two endpoints.

The assumption that $\rho \in (0, 1)$ is fixed is essential to the stated asymptotic conclusion. At the lower endpoint, $c_\rho \to \infty$ as $\rho \downarrow 0$. Indeed, fix $C > 0$. Dominated convergence in (30) gives

$$\kappa_\rho(C) \longrightarrow \int_0^\infty e^{-s^2/8} \cos(Cs)\, ds = \sqrt{2\pi} e^{-2C^2} > 0.$$

The sign relation in Lemma 10 therefore implies $c_\rho > C$ for all sufficiently small $\rho > 0$. Since $C > 0$ is arbitrary, it follows that $c_\rho \to \infty$. At the other endpoint, $c_\rho \to 0$ as $\rho \uparrow 1$. To see this, again fix $C > 0$. Dominated convergence in (30) gives

$$\kappa_\rho(C) \longrightarrow \kappa_1(C) = -\int_0^\infty s e^{-s^2/8} \sin(Cs)\, ds = -4C\sqrt{2\pi} e^{-2C^2} < 0.$$

The same sign relation implies $0 < c_\rho < C$ whenever $\rho < 1$ is sufficiently close to one. Since $C > 0$ is arbitrary, the claim follows.

Acknowledgments: The authors used OpenAI's GPT-5.6 Sol to assist in the derivations in Sections 5 and 6. The authors independently verified all arguments and take full responsibility for the manuscript.

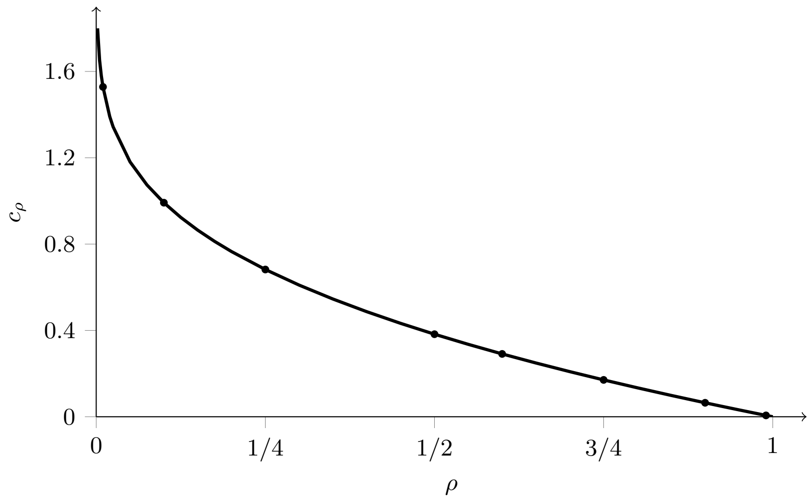

Figure 2: The dependence of c_ρ on ρ. The dots indicate the values reported in the preceding table. The curve was obtained numerically by solving $D_\rho(-2c_\rho) = 0$. It is truncated near $\rho = 0$, where $c_\rho \to \infty$, and uses the continuous extension $c_1 = 0$.

References

Yu Hin Au and Levent Tunçel. Elementary polytopes with high lift-and-project ranks for strong positive semidefinite operators. *Discrete Optimization*, 27:103–129, 2018. doi: 10.1016/j.disopt.2017.10.001.

Alexander Bockmayr, Friedrich Eisenbrand, Mark Hartmann, and Andreas S. Schulz. On the Chvátal rank of polytopes in the 0/1 cube. *Discrete Applied Mathematics*, 98(1–2):21–27, 1999. doi: 10.1016/S0166-218X(99)00156-0.

Kevin K. H. Cheung. Computation of the Lasserre ranks of some polytopes. *Mathematics of Operations Research*, 32(1):88–94, 2007. doi: 10.1287/moor.1060.0212.

Dan Dai and Roderick Wong. Global asymptotics of Krawtchouk polynomials—a Riemann–Hilbert approach. *Chinese Annals of Mathematics, Series B*, 28(1):1–34, 2007. doi: 10.1007/s11401-006-0195-3.

Diego Dominici. Asymptotic analysis of the Krawtchouk polynomials by the WKB method. *The Ramanujan Journal*, 15(3):303–338, 2008. doi: 10.1007/s11139-007-9078-9.

Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher Transcendental Functions*, volume 2. McGraw–Hill, New York, 1953. Based, in part, on notes left by Harry Bateman.

Mourad E. H. Ismail and Plamen Simeonov. Strong asymptotics for Krawtchouk polynomials. *Journal of Computational and Applied Mathematics*, 100(2):121–144, 1998. doi: 10.1016/S0377-0427(98)00183-6.

Kurt Johansson. Non-intersecting paths, random tilings and random matrices. *Probability Theory and Related Fields*, 123:225–280, 2002. doi: 10.1007/s004400100187.

Mykhailo Krawtchouk. Sur une généralisation des polynômes d'Hermite. *Comptes Rendus de l'Académie des Sciences*, 189:620–622, 1929.

Adam Kurpisz. Sum-of-squares bounds via boolean function analysis. In *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, volume 132 of *LIPIcs*, pages 79:1–79:15. Schloss Dagstuhl, 2019. doi: 10.4230/LIPIcs.ICALP.2019.79.

Adam Kurpisz and Elias Samuel Wirth. Sum of squares bounds for the empty integral hull problem. In *Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation*, pages 443–451. ACM, 2023. doi: 10.1145/3597066.3597098.

Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. Sum-of-squares hierarchy lower bounds for symmetric formulations. In *Integer Programming and Combinatorial Optimization (IPCO 2016)*, volume 9682 of *Lecture Notes in Computer Science*, pages 362–374. Springer, 2016a. doi: 10.1007/978-3-319-33461-5_30.

Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. Tight sum-of-squares lower bounds for binary polynomial optimization problems. In *43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016)*, volume 55 of *LIPIcs*, pages 78:1–78:14. Schloss Dagstuhl, 2016b. doi: 10.4230/LIPIcs.ICALP.2016.78.

Adam Kurpisz, Aaron Potechin, and Elias Wirth. SoS certification for symmetric quadratic functions and its connection to constrained boolean hypercube optimization. *Mathematics of Operations Research*, 51(1):255–276, 2026. doi: 10.1287/moor.2024.0422. Published online February 6, 2025.

Jean B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. In Karen Aardal and Bert Gerards, editors, *Integer Programming and Combinatorial Optimization (IPCO 2001)*, volume 2081 of *Lecture Notes in Computer Science*, pages 293–303. Springer, 2001. doi: 10.1007/3-540-45535-3_23.

Monique Laurent. A comparison of the Sherali–Adams, Lovász–Schrijver, and Lasserre relaxations for 0-1 programming. *Mathematics of Operations Research*, 28(3):470–496, 2003. doi: 10.1287/moor.28.3.470.16391.

X.-C. Li and Roderick Wong. A uniform asymptotic expansion for Krawtchouk polynomials. *Journal of Approximation Theory*, 106(1):155–184, 2000. doi: 10.1006/jath.2000.3474.

Florence Jessie MacWilliams and Neil James Alexander Sloane. *The Theory of Error-Correcting Codes*, volume 16 of *North-Holland Mathematical Library*. North-Holland, Amsterdam, 1977. ISBN 978-0-444-85009-6.

Wei-Yuan Qiu and Roderick Wong. Asymptotic expansion of the Krawtchouk polynomials and their zeros. *Computational Methods and Function Theory*, 4(1):189–226, 2004. doi: 10.1007/BF03321065.

Lucas Slot and Monique Laurent. Sum-of-squares hierarchies for binary polynomial optimization. In Mohit Singh and David Williamson, editors, *Integer Programming and Combinatorial Optimization (IPCO 2021)*, volume 12707 of *Lecture Notes in Computer Science*, pages 43–57. Springer, 2021. doi: 10.1007/978-3-030-73879-24.

Riaz A. Usmani. Inversion of a tridiagonal Jacobi matrix. *Linear Algebra and Its Applications*, 212–213:413–414, 1994. doi: 10.1016/0024-3795(94)90414-6.